# 8.8 Improper Integrals

- **Evaluate an improper integral that has an infinite limit of integration.**
- **Evaluate an improper integral that has an infinite discontinuity.**

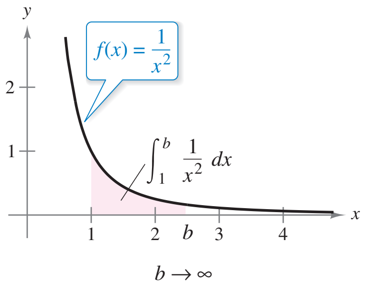
## Improper Integrals with Infinite Limits of Integration

The definition of a definite integral

$$\int_a^b f(x)\,dx$$

requires that the interval $[a, b]$ be finite. Furthermore, the Fundamental Theorem of Calculus, by which you have been evaluating definite integrals, requires that $f$ be continuous on $[a, b]$. In this section, you will study a procedure for evaluating integrals that do not satisfy these requirements—usually because either one or both of the limits of integration are infinite, or because $f$ has a finite number of infinite discontinuities in the interval $[a, b]$. Integrals that possess either property are **improper integrals.** Note that a function $f$ is said to have an **infinite discontinuity** at $c$ when, *from the right or left,*

$$\lim_{x\to c} f(x) = \infty \quad \text{or} \quad \lim_{x\to c} f(x) = -\infty.$$




The unbounded region has an area of 1.
**Figure 8.17**

To get an idea of how to evaluate an improper integral, consider the integral

$$\int_1^b \frac{dx}{x^2} = -\frac{1}{x}\Big]_1^b = -\frac{1}{b} + 1 = 1 - \frac{1}{b}$$

which can be interpreted as the area of the shaded region shown in Figure 8.17. Taking the limit as $b \to \infty$ produces

$$\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty}\left(\int_1^b \frac{dx}{x^2}\right) = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1.$$

This improper integral can be interpreted as the area of the *unbounded* region between the graph of $f(x) = 1/x^2$ and the $x$-axis (to the right of $x = 1$).

### Definition of Improper Integrals with Infinite Integration Limits

**1.** If $f$ is continuous on the interval $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

**2.** If $f$ is continuous on the interval $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

**3.** If $f$ is continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

where $c$ is any real number (see Exercise 111).

In the first two cases, the improper integral **converges** when the limit exists—otherwise, the improper integral **diverges**. In the third case, the improper integral on the left diverges when either of the improper integrals on the right diverges.

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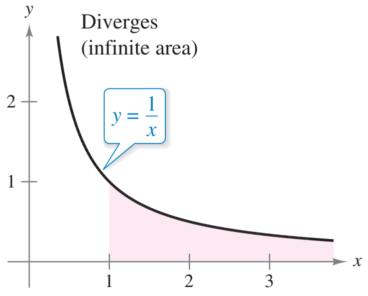



This unbounded region has an infinite area.
**Figure 8.18**

## EXAMPLE 1 An Improper Integral That Diverges

Evaluate $\int_1^{\infty} \frac{dx}{x}$.

### Solution

$$\int_1^{\infty} \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} \qquad \text{Take limit as } b \to \infty.$$

$$= \lim_{b\to\infty} \Big[\ln x\Big]_1^b \qquad \text{Apply Log Rule.}$$

$$= \lim_{b\to\infty} (\ln b - 0) \qquad \text{Apply Fundamental Theorem of Calculus.}$$

$$= \infty \qquad \text{Evaluate limit.}$$

The limit does not exist. So, you can conclude that the improper integral diverges. See Figure 8.18.

Try comparing the regions shown in Figures 8.17 and 8.18. They look similar, yet the region in Figure 8.17 has a finite area of 1 and the region in Figure 8.18 has an infinite area.

## EXAMPLE 2 Improper Integrals That Converge

Evaluate each improper integral.

**a.** $\int_0^{\infty} e^{-x}\, dx$

**b.** $\int_0^{\infty} \frac{1}{x^2+1}\, dx$

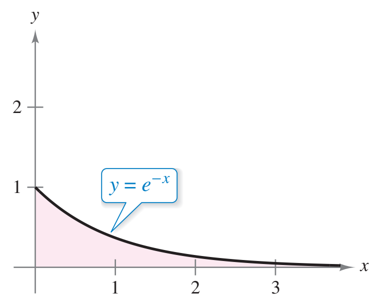
### Solution

**a.**
$$\int_0^{\infty} e^{-x}\, dx = \lim_{b\to\infty} \int_0^b e^{-x}\, dx$$
$$= \lim_{b\to\infty} \Big[-e^{-x}\Big]_0^b$$
$$= \lim_{b\to\infty} (-e^{-b} + 1)$$
$$= 1$$

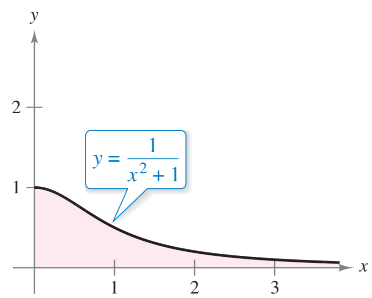
See Figure 8.19.

**b.**
$$\int_0^{\infty} \frac{1}{x^2+1}\, dx = \lim_{b\to\infty} \int_0^{\infty} \frac{1}{x^2+1}\, dx$$
$$= \lim_{b\to\infty} \Big[\arctan x\Big]_0^b$$
$$= \lim_{b\to\infty} \arctan b$$
$$= \frac{\pi}{2}$$

See Figure 8.20.




The area of the unbounded region is 1.
**Figure 8.19**




The area of the unbounded region is $\pi/2$.
**Figure 8.20**

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In the next example, note how L'Hôpital's Rule can be used to evaluate an improper integral.

### EXAMPLE 3 Using L'Hôpital's Rule with an Improper Integral

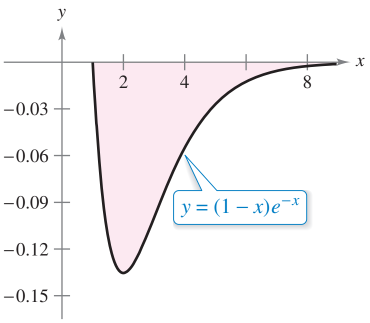
Evaluate $\int_1^{\infty} (1 - x)e^{-x}\,dx$.

**Solution** Use integration by parts, with $dv = e^{-x}\,dx$ and $u = (1 - x)$.

$$\begin{aligned}\int (1 - x)e^{-x}\,dx &= -e^{-x}(1 - x) - \int e^{-x}\,dx\\ &= -e^{-x} + xe^{-x} + e^{-x} + C\\ &= xe^{-x} + C\end{aligned}$$

Now, apply the definition of an improper integral.

$$\begin{aligned}\int_1^{\infty} (1 - x)e^{-x}\,dx &= \lim_{b\to\infty}\Big[xe^{-x}\Big]_1^b\\ &= \lim_{b\to\infty}\left(\frac{b}{e^b} - \frac{1}{e}\right)\\ &= \lim_{b\to\infty}\frac{b}{e^b} - \lim_{b\to\infty}\frac{1}{e}\end{aligned}$$




The area of the unbounded region is $|-1/e|$.
**Figure 8.21**

For the first limit, use L'Hôpital's Rule.

$$\lim_{b\to\infty}\frac{b}{e^b} = \lim_{b\to\infty}\frac{1}{e^b} = 0$$

So, you can conclude that

$$\begin{aligned}\int_1^{\infty} (1 - x)e^{-x}\,dx &= \lim_{b\to\infty}\frac{b}{e^b} - \lim_{b\to\infty}\frac{1}{e}\\ &= 0 - \frac{1}{e}\\ &= -\frac{1}{e}.\end{aligned}$$

See Figure 8.21.

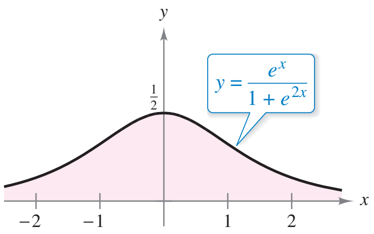
### EXAMPLE 4 Infinite Upper and Lower Limits of Integration

Evaluate $\int_{-\infty}^{\infty} \frac{e^x}{1 + e^{2x}}\,dx$.

**Solution** Note that the integrand is continuous on $(-\infty, \infty)$. To evaluate the integral, you can break it into two parts, choosing $c = 0$ as a convenient value.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^x}{1 + e^{2x}}\,dx &= \int_{-\infty}^{0} \frac{e^x}{1 + e^{2x}}\,dx + \int_{0}^{\infty} \frac{e^x}{1 + e^{2x}}\,dx\\ &= \lim_{b\to-\infty}\Big[\arctan e^x\Big]_b^0 + \lim_{b\to\infty}\Big[\arctan e^x\Big]_0^b\\ &= \lim_{b\to-\infty}\left(\frac{\pi}{4} - \arctan e^b\right) + \lim_{b\to\infty}\left(\arctan e^b - \frac{\pi}{4}\right)\\ &= \frac{\pi}{4} - 0 + \frac{\pi}{2} - \frac{\pi}{4}\\ &= \frac{\pi}{2}\end{aligned}$$

See Figure 8.22.




The area of the unbounded region is $\pi/2$.
**Figure 8.22**

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The work required to move a 15-metric-ton space module an unlimited distance away from Earth is about $6.984 \times 10^{11}$ foot-pounds.

**EXAMPLE 5** **Sending a Space Module into Orbit**

In Example 3 in Section 7.5, you found that it would require 10,000 mile-tons of work to propel a 15-metric-ton space module to a height of 800 miles above Earth. How much work is required to propel the module an unlimited distance away from Earth's surface?

**Solution** At first you might think that an infinite amount of work would be required. But if this were the case, it would be impossible to send rockets into outer space. Because this has been done, the work required must be finite. You can determine the work in the following manner. Using the integral in Example 3, Section 7.5, replace the upper bound of 4800 miles by $\infty$ and write

$$\begin{aligned}
W &= \int_{4000}^{\infty} \frac{240{,}000{,}000}{x^2}\,dx \\
&= \lim_{b\to\infty}\left[-\frac{240{,}000{,}000}{x}\right]_{4000}^{b} \\
&= \lim_{b\to\infty}\left(-\frac{240{,}000{,}000}{b} + \frac{240{,}000{,}000}{4000}\right) \\
&= 60{,}000 \text{ mile-tons} \\
&\approx 6.984 \times 10^{11} \text{ foot-pounds.}
\end{aligned}$$

In SI units, using a conversion factor of

$$1 \text{ foot-pound} \approx 1.35582 \text{ joules}$$

the work done is $W \approx 9.469 \times 10^{11}$ joules.

## Improper Integrals with Infinite Discontinuities

The second basic type of improper integral is one that has an infinite discontinuity *at or between* the limits of integration.

**Definition of Improper Integrals with Infinite Discontinuities**

1. If $f$ is continuous on the interval $[a, b)$ and has an infinite discontinuity at $b$, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$

2. If $f$ is continuous on the interval $(a, b]$ and has an infinite discontinuity at $a$, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$

3. If $f$ is continuous on the interval $[a, b]$, except for some $c$ in $(a, b)$ at which $f$ has an infinite discontinuity, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In the first two cases, the improper integral **converges** when the limit exists—otherwise, the improper integral **diverges**. In the third case, the improper integral on the left diverges when either of the improper integrals on the right diverges.

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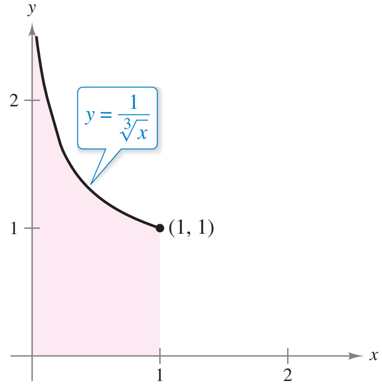
## EXAMPLE 6 An Improper Integral with an Infinite Discontinuity

Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}}$.

**Solution** The integrand has an infinite discontinuity at $x = 0$, as shown in Figure 8.23. You can evaluate this integral as shown below.

$$\int_0^1 x^{-1/3}\,dx = \lim_{b\to 0^+}\left[\frac{x^{2/3}}{2/3}\right]_b^1 = \lim_{b\to 0^+}\frac{3}{2}(1 - b^{2/3}) = \frac{3}{2}$$




Infinite discontinuity at $x = 0$

**Figure 8.23**

## EXAMPLE 7 An Improper Integral That Diverges

Evaluate $\displaystyle\int_0^2 \frac{dx}{x^3}$.

**Solution** Because the integrand has an infinite discontinuity at $x = 0$, you can write

$$\int_0^2 \frac{dx}{x^3} = \lim_{b\to 0^+}\left[-\frac{1}{2x^2}\right]_b^2 = \lim_{b\to 0^+}\left(-\frac{1}{8} + \frac{1}{2b^2}\right) = \infty.$$

So, you can conclude that the improper integral diverges.

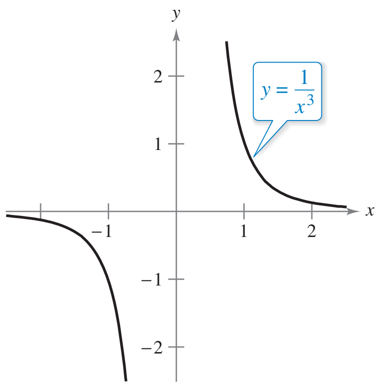
## EXAMPLE 8 An Improper Integral with an Interior Discontinuity

Evaluate $\displaystyle\int_{-1}^2 \frac{dx}{x^3}$.

**Solution** This integral is improper because the integrand has an infinite discontinuity at the interior point $x = 0$, as shown in Figure 8.24. So, you can write

$$\int_{-1}^2 \frac{dx}{x^3} = \int_{-1}^0 \frac{dx}{x^3} + \int_0^2 \frac{dx}{x^3}.$$

From Example 7, you know that the second integral diverges. So, the original improper integral also diverges. ■




The improper integral $\displaystyle\int_{-1}^2 \frac{dx}{x^3}$ diverges.

**Figure 8.24**

Remember to check for infinite discontinuities at interior points as well as at endpoints when determining whether an integral is improper. For instance, if you had not recognized that the integral in Example 8 was improper, you would have obtained the *incorrect* result

$$\int_{-1}^2 \frac{dx}{x^3} = \left[\frac{-1}{2x^2}\right]_{-1}^2 = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}.$$

Incorrect evaluation

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The integral in the next example is improper for *two* reasons. One limit of integration is infinite, and the integrand has an infinite discontinuity at the outer limit of integration.

## EXAMPLE 9 A Doubly Improper Integral

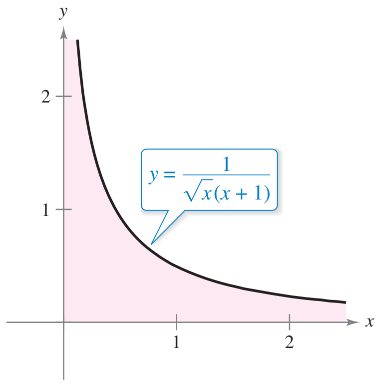
See LarsonCalculus.com for an interactive version of this type of example.

Evaluate $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)}$.

**Solution** To evaluate this integral, split it at a convenient point (say, $x = 1$) and write

$$\begin{aligned}\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)} &= \int_0^{1} \frac{dx}{\sqrt{x}(x+1)} + \int_1^{\infty} \frac{dx}{\sqrt{x}(x+1)}\\ &= \lim_{b\to 0^+}\Big[2\arctan\sqrt{x}\Big]_b^1 + \lim_{c\to\infty}\Big[2\arctan\sqrt{x}\Big]_1^c\\ &= \lim_{b\to 0^+}\left(2\arctan 1 - 2\arctan\sqrt{b}\right) + \lim_{c\to\infty}\left(2\arctan\sqrt{c} - 2\arctan 1\right)\\ &= 2\left(\frac{\pi}{4}\right) - 0 + 2\left(\frac{\pi}{2}\right) - 2\left(\frac{\pi}{4}\right)\\ &= \pi.\end{aligned}$$

See Figure 8.25.




The area of the unbounded region is $\pi$.
**Figure 8.25**

## EXAMPLE 10 An Application Involving Arc Length

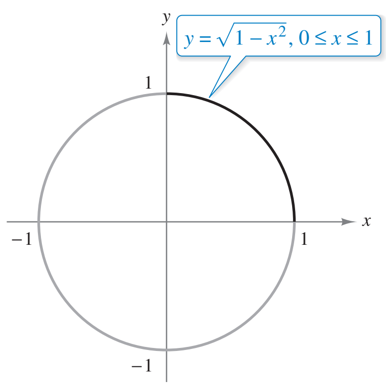
Use the formula for arc length to show that the circumference of the circle $x^2 + y^2 = 1$ is $2\pi$.

**Solution** To simplify the work, consider the quarter circle given by $y = \sqrt{1-x^2}$, where $0 \le x \le 1$. The function $y$ is differentiable for any $x$ in this interval except $x = 1$. Therefore, the arc length of the quarter circle is given by the improper integral

$$\begin{aligned}s &= \int_0^1 \sqrt{1+(y')^2}\,dx\\ &= \int_0^1 \sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,dx\\ &= \int_0^1 \frac{dx}{\sqrt{1-x^2}}.\end{aligned}$$

This integral is improper because it has an infinite discontinuity at $x = 1$. So, you can write

$$\begin{aligned}s &= \int_0^1 \frac{dx}{\sqrt{1-x^2}}\\ &= \lim_{b\to 1^-}\Big[\arcsin x\Big]_0^b\\ &= \lim_{b\to 1^-}\left(\arcsin b - \arcsin 0\right)\\ &= \frac{\pi}{2} - 0\\ &= \frac{\pi}{2}.\end{aligned}$$




The circumference of the circle is $2\pi$.
**Figure 8.26**

Finally, multiplying by 4, you can conclude that the circumference of the circle is $4s = 2\pi$, as shown in Figure 8.26.

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This section concludes with a useful theorem describing the convergence or divergence of a common type of improper integral. The proof of this theorem is left as an exercise (see Exercise 49).

**THEOREM 8.5 A Special Type of Improper Integral**

$$\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1}, & p > 1 \\ \text{diverges}, & p \le 1 \end{cases}$$

## EXAMPLE 11 An Application Involving a Solid of Revolution

The solid formed by revolving (about the $x$-axis) the *unbounded* region lying between the graph of $f(x) = 1/x$ and the $x$-axis $(x \ge 1)$ is called **Gabriel's Horn.** (See Figure 8.27.) Show that this solid has a finite volume and an infinite surface area.

**Solution** Using the disk method and Theorem 8.5, you can determine the volume to be

$$V = \pi \int_1^{\infty} \left(\frac{1}{x}\right)^2 dx \qquad \text{Theorem 8.5, } p = 2 > 1$$

$$= \pi\left(\frac{1}{2-1}\right)$$

$$= \pi.$$

The surface area is given by

$$S = 2\pi \int_1^{\infty} f(x)\sqrt{1 + [f'(x)]^2}\, dx = 2\pi \int_1^{\infty} \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\, dx.$$

Because

$$\sqrt{1 + \frac{1}{x^4}} > 1$$

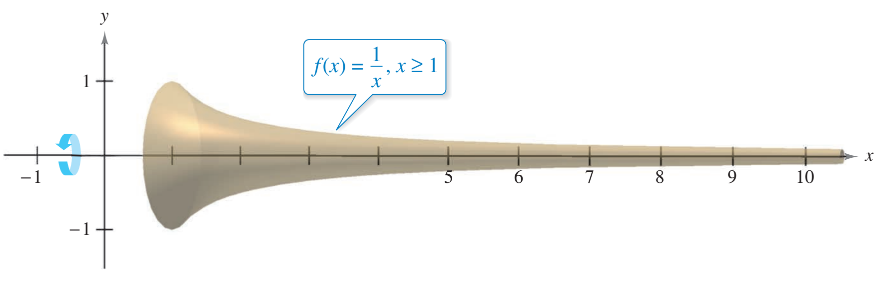
on the interval $[1, \infty)$, and the improper integral

$$\int_1^{\infty} \frac{1}{x}\, dx$$

diverges, you can conclude that the improper integral

$$\int_1^{\infty} \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\, dx$$

also diverges. (See Exercise 52.) So, the surface area is infinite.




Gabriel's Horn has a finite volume and an infinite surface area.
**Figure 8.27**

**FOR FURTHER INFORMATION** For further investigation of solids that have finite volumes and infinite surface areas, see the article "Supersolids: Solids Having Finite Volume and Infinite Surfaces" by William P. Love in *Mathematics Teacher*. To view this article, go to *MathArticles.com.*

**FOR FURTHER INFORMATION** To learn about another function that has a finite volume and an infinite surface area, see the article "Gabriel's Wedding Cake" by Julian F. Fleron in *The College Mathematics Journal.* To view this article, go to *MathArticles.com.*

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# 8.8 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

**Determining Whether an Integral Is Improper** In Exercises 1–8, decide whether the integral is improper. Explain your reasoning.

1. $\displaystyle\int_0^1 \frac{dx}{5x - 3}$

2. $\displaystyle\int_1^2 \frac{dx}{x^3}$

3. $\displaystyle\int_0^1 \frac{2x - 5}{x^2 - 5x + 6}\, dx$

4. $\displaystyle\int_1^\infty \ln(x^2)\, dx$

5. $\displaystyle\int_0^2 e^{-x}\, dx$

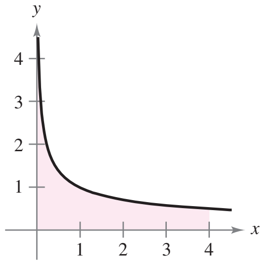
6. $\displaystyle\int_0^\infty \cos x\, dx$

7. $\displaystyle\int_{-\infty}^\infty \frac{\sin x}{4 + x^2}\, dx$

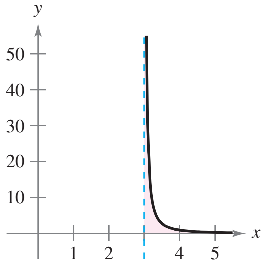
8. $\displaystyle\int_0^{\pi/4} \csc x\, dx$

**Evaluating an Improper Integral** In Exercises 9–12, explain why the integral is improper and determine whether it diverges or converges. Evaluate the integral if it converges.

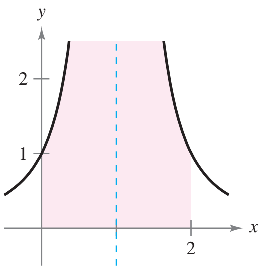
9. $\displaystyle\int_0^4 \frac{1}{\sqrt{x}}\, dx$



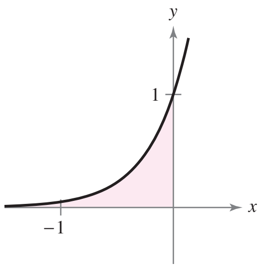

10. $\displaystyle\int_3^4 \frac{1}{(x - 3)^{3/2}}\, dx$




11. $\displaystyle\int_0^2 \frac{1}{(x - 1)^2}\, dx$




12. $\displaystyle\int_{-\infty}^0 e^{3x}\, dx$




**Writing** In Exercises 13–16, explain why the evaluation of the integral is *incorrect*. Use the integration capabilities of a graphing utility to attempt to evaluate the integral. Determine whether the utility gives the correct answer.

13. $\displaystyle\int_{-1}^1 \frac{1}{x^2}\, dx = -2$

14. $\displaystyle\int_{-2}^2 \frac{-2}{(x - 1)^3}\, dx = \frac{8}{9}$

15. $\displaystyle\int_0^\infty e^{-x}\, dx = 0$

16. $\displaystyle\int_0^\pi \sec x\, dx = 0$

**Evaluating an Improper Integral** In Exercises 17–32, determine whether the improper integral diverges or converges. Evaluate the integral if it converges.

17. $\displaystyle\int_1^\infty \frac{1}{x^3}\, dx$

18. $\displaystyle\int_1^\infty \frac{6}{x^4}\, dx$

19. $\displaystyle\int_1^\infty \frac{3}{\sqrt[3]{x}}\, dx$

20. $\displaystyle\int_1^\infty \frac{4}{\sqrt[4]{x}}\, dx$

21. $\displaystyle\int_{-\infty}^0 xe^{-4x}\, dx$

22. $\displaystyle\int_0^\infty xe^{-x/3}\, dx$

23. $\displaystyle\int_0^\infty x^2e^{-x}\, dx$

24. $\displaystyle\int_0^\infty e^{-x}\cos x\, dx$

25. $\displaystyle\int_4^\infty \frac{1}{x(\ln x)^3}\, dx$

26. $\displaystyle\int_1^\infty \frac{\ln x}{x}\, dx$

27. $\displaystyle\int_{-\infty}^\infty \frac{4}{16 + x^2}\, dx$

28. $\displaystyle\int_0^\infty \frac{x^3}{(x^2 + 1)^2}\, dx$

29. $\displaystyle\int_0^\infty \frac{1}{e^x + e^{-x}}\, dx$

30. $\displaystyle\int_0^\infty \frac{e^x}{1 + e^x}\, dx$

31. $\displaystyle\int_0^\infty \cos \pi x\, dx$

32. $\displaystyle\int_0^\infty \sin \frac{x}{2}\, dx$

**Evaluating an Improper Integral** In Exercises 33–48, determine whether the improper integral diverges or converges. Evaluate the integral if it converges, and check your results with the results obtained by using the integration capabilities of a graphing utility.

33. $\displaystyle\int_0^1 \frac{1}{x^2}\, dx$

34. $\displaystyle\int_0^5 \frac{10}{x}\, dx$

35. $\displaystyle\int_0^2 \frac{1}{\sqrt[3]{x - 1}}\, dx$

36. $\displaystyle\int_0^8 \frac{3}{\sqrt{8 - x}}\, dx$

37. $\displaystyle\int_0^1 x \ln x\, dx$

38. $\displaystyle\int_0^e \ln x^2\, dx$

39. $\displaystyle\int_0^{\pi/2} \tan \theta\, d\theta$

40. $\displaystyle\int_0^{\pi/2} \sec \theta\, d\theta$

41. $\displaystyle\int_2^4 \frac{2}{x\sqrt{x^2 - 4}}\, dx$

42. $\displaystyle\int_3^6 \frac{1}{\sqrt{36 - x^2}}\, dx$

43. $\displaystyle\int_3^5 \frac{1}{\sqrt{x^2 - 9}}\, dx$

44. $\displaystyle\int_0^5 \frac{1}{25 - x^2}\, dx$

45. $\displaystyle\int_3^\infty \frac{1}{x\sqrt{x^2 - 9}}\, dx$

46. $\displaystyle\int_4^\infty \frac{\sqrt{x^2 - 16}}{x^2}\, dx$

47. $\displaystyle\int_0^\infty \frac{4}{\sqrt{x}(x + 6)}\, dx$

48. $\displaystyle\int_1^\infty \frac{1}{x \ln x}\, dx$

**Finding Values** In Exercises 49 and 50, determine all values of $p$ for which the improper integral converges.

49. $\displaystyle\int_1^\infty \frac{1}{x^p}\, dx$

50. $\displaystyle\int_0^1 \frac{1}{x^p}\, dx$

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**51. Mathematical Induction** Use mathematical induction to verify that the following integral converges for any positive integer $n$.

$$\int_0^\infty x^n e^{-x}\,dx$$

**52. Comparison Test for Improper Integrals** In some cases, it is impossible to find the exact value of an improper integral, but it is important to determine whether the integral converges or diverges. Suppose the functions $f$ and $g$ are continuous and $0 \le g(x) \le f(x)$ on the interval $[a, \infty)$. It can be shown that if $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ also converges, and if $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ also diverges. This is known as the Comparison Test for improper integrals.

(a) Use the Comparison Test to determine whether $\int_1^\infty e^{-x^2}\,dx$ converges or diverges. (*Hint:* Use the fact that $e^{-x^2} \le e^{-x}$ for $x \ge 1$.)

(b) Use the Comparison Test to determine whether $\displaystyle\int_1^\infty \frac{1}{x^5+1}\,dx$ converges or diverges. (*Hint:* Use the fact that $\dfrac{1}{x^5+1} \le \dfrac{1}{x^5}$ for $x \ge 1$.)

**Convergence or Divergence** **In Exercises 53–62, use the results of Exercises 49–52 to determine whether the improper integral converges or diverges.**

**53.** $\displaystyle\int_0^1 \frac{1}{x^5}\,dx$

**54.** $\displaystyle\int_0^1 \frac{1}{\sqrt[5]{x}}\,dx$

**55.** $\displaystyle\int_1^\infty \frac{1}{x^5}\,dx$

**56.** $\displaystyle\int_0^\infty x^4 e^{-x}\,dx$

**57.** $\displaystyle\int_1^\infty \frac{1}{x^2+5}\,dx$

**58.** $\displaystyle\int_2^\infty \frac{1}{\sqrt{x-1}}\,dx$

**59.** $\displaystyle\int_2^\infty \frac{1}{\sqrt[3]{x(x-1)}}\,dx$

**60.** $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}(x+1)}\,dx$

**61.** $\displaystyle\int_1^\infty \frac{1-\sin x}{x^2}\,dx$

**62.** $\displaystyle\int_0^\infty \frac{1}{e^x+x}\,dx$

## WRITING ABOUT CONCEPTS

**63. Improper Integrals** Describe the different types of improper integrals.

**64. Improper Integrals** Define the terms *converges* and *diverges* when working with improper integrals.

**65. Improper Integral** Explain why $\displaystyle\int_{-1}^1 \frac{1}{x^3}\,dx \ne 0$.

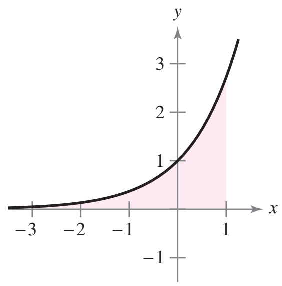
**66. Improper Integral** Consider the integral

$$\int_0^3 \frac{10}{x^2-2x}\,dx.$$

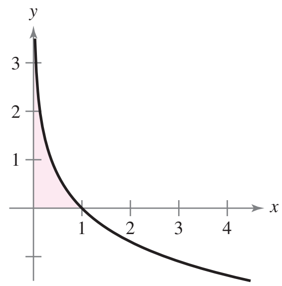
To determine the convergence or divergence of the integral, how many improper integrals must be analyzed? What must be true of each of these integrals if the given integral converges?

**Area** **In Exercises 67–70, find the area of the unbounded shaded region.**

**67.** $y = e^x, \quad -\infty < x \le 1$

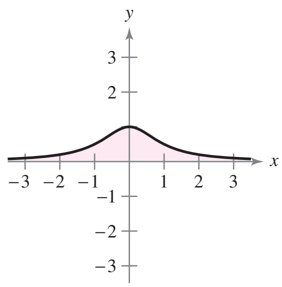


**68.** $y = -\ln x$



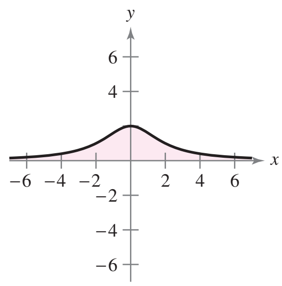
**69.** Witch of Agnesi:

$$y = \frac{1}{x^2+1}$$



**70.** Witch of Agnesi:

$$y = \frac{8}{x^2+4}$$



**Area and Volume** **In Exercises 71 and 72, consider the region satisfying the inequalities. (a) Find the area of the region. (b) Find the volume of the solid generated by revolving the region about the $x$-axis. (c) Find the volume of the solid generated by revolving the region about the $y$-axis.**

**71.** $y \le e^{-x},\ y \ge 0,\ x \ge 0$

**72.** $y \le \dfrac{1}{x^2},\ y \ge 0,\ x \ge 1$

**73. Arc Length** Sketch the graph of the hypocycloid of four cusps $x^{2/3} + y^{2/3} = 4$ and find its perimeter.

**74. Arc Length** Find the arc length of the graph of $y = \sqrt{16 - x^2}$ over the interval $[0, 4]$.

**75. Surface Area** The region bounded by $(x-2)^2 + y^2 = 1$ is revolved about the $y$-axis to form a torus. Find the surface area of the torus.

**76. Surface Area** Find the area of the surface formed by revolving the graph of $y = 2e^{-x}$ on the interval $[0, \infty)$ about the $x$-axis.

**Propulsion** **In Exercises 77 and 78, use the weight of the rocket to answer each question. (Use 4000 miles as the radius of Earth and do not consider the effect of air resistance.)**

**(a) How much work is required to propel the rocket an unlimited distance away from Earth's surface?**

**(b) How far has the rocket traveled when half the total work has occurred?**

**77.** 5-ton rocket

**78.** 10-ton rocket

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**Probability** **A nonnegative function $f$ is called a *probability density function* if**

$$\int_{-\infty}^{\infty} f(t)\,dt = 1.$$

**The probability that $x$ lies between $a$ and $b$ is given by**

$$P(a \le x \le b) = \int_a^b f(t)\,dt.$$

**The expected value of $x$ is given by**

$$E(x) = \int_{-\infty}^{\infty} t f(t)\,dt.$$

**In Exercises 79 and 80, (a) show that the nonnegative function is a probability density function, (b) find $P(0 \le x \le 4)$, and (c) find $E(x)$.**

**79.** $f(t) = \begin{cases} \frac{1}{7}e^{-t/7}, & t \ge 0 \\ 0, & t < 0 \end{cases}$ **80.** $f(t) = \begin{cases} \frac{2}{5}e^{-2t/5}, & t \ge 0 \\ 0, & t < 0 \end{cases}$

**Capitalized Cost** **In Exercises 81 and 82, find the capitalized cost $C$ of an asset (a) for $n = 5$ years, (b) for $n = 10$ years, and (c) forever. The capitalized cost is given by**

$$C = C_0 + \int_0^n c(t)e^{-rt}\,dt$$

**where $C_0$ is the original investment, $t$ is the time in years, $r$ is the annual interest rate compounded continuously, and $c(t)$ is the annual cost of maintenance.**

**81.** $C_0 = \$650{,}000$
$c(t) = \$25{,}000$
$r = 0.06$

**82.** $C_0 = \$650{,}000$
$c(t) = \$25{,}000(1 + 0.08t)$
$r = 0.06$

**83. Electromagnetic Theory** The magnetic potential $P$ at a point on the axis of a circular coil is given by

$$P = \frac{2\pi NIr}{k}\int_c^{\infty} \frac{1}{(r^2 + x^2)^{3/2}}\,dx$$

where $N$, $I$, $r$, $k$, and $c$ are constants. Find $P$.

**84. Gravitational Force** A "semi-infinite" uniform rod occupies the nonnegative $x$-axis. The rod has a linear density $\delta$, which means that a segment of length $dx$ has a mass of $\delta\,dx$. A particle of mass $M$ is located at the point $(-a, 0)$. The gravitational force $F$ that the rod exerts on the mass is given by

$$F = \int_0^{\infty} \frac{GM\delta}{(a + x)^2}\,dx$$

where $G$ is the gravitational constant. Find $F$.

**True or False?** **In Exercises 85–88, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

**85.** If $f$ is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_0^{\infty} f(x)\,dx$ converges.

**86.** If $f$ is continuous on $[0, \infty)$ and $\int_0^{\infty} f(x)\,dx$ diverges, then $\lim_{x\to\infty} f(x) \ne 0$.

**87.** If $f'$ is continuous on $[0, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then

$$\int_0^{\infty} f'(x)\,dx = -f(0).$$

**88.** If the graph of $f$ is symmetric with respect to the origin or the $y$-axis, then $\int_0^{\infty} f(x)\,dx$ converges if and only if $\int_{-\infty}^{\infty} f(x)\,dx$ converges.

**89. Comparing Integrals**

(a) Show that $\int_{-\infty}^{\infty} \sin x\,dx$ diverges.

(b) Show that $\lim_{a\to\infty} \int_{-a}^{a} \sin x\,dx = 0$.

(c) What do parts (a) and (b) show about the definition of improper integrals?

**90. Making an Integral Improper** For each integral, find a nonnegative real number $b$ that makes the integral improper. Explain your reasoning.

(a) $\displaystyle\int_0^b \frac{1}{x^2 - 9}\,dx$ (b) $\displaystyle\int_0^b \frac{1}{\sqrt{4 - x}}\,dx$

(c) $\displaystyle\int_0^b \frac{x}{x^2 - 7x + 12}\,dx$ (d) $\displaystyle\int_b^{10} \ln x\,dx$

(e) $\displaystyle\int_0^b \tan 2x\,dx$ (f) $\displaystyle\int_0^b \frac{\cos x}{1 - \sin x}\,dx$

**91. Writing**

(a) The improper integrals

$$\int_1^{\infty} \frac{1}{x}\,dx \quad \text{and} \quad \int_1^{\infty} \frac{1}{x^2}\,dx$$

diverge and converge, respectively. Describe the essential differences between the integrands that cause one integral to converge and the other to diverge.

(b) Sketch a graph of the function $y = (\sin x)/x$ over the interval $(1, \infty)$. Use your knowledge of the definite integral to make an inference as to whether the integral

$$\int_1^{\infty} \frac{\sin x}{x}\,dx$$

converges. Give reasons for your answer.

(c) Use one iteration of integration by parts on the integral in part (b) to determine its divergence or convergence.

**92. Exploration** Consider the integral

$$\int_0^{\pi/2} \frac{4}{1 + (\tan x)^n}\,dx$$

where $n$ is a positive integer.

(a) Is the integral improper? Explain.

(b) Use a graphing utility to graph the integrand for $n = 2, 4, 8,$ and $12$.

(c) Use the graphs to approximate the integral as $n \to \infty$.

(d) Use a computer algebra system to evaluate the integral for the values of $n$ in part (b). Make a conjecture about the value of the integral for any positive integer $n$. Compare your results with your answer in part (c).

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**93. Normal Probability** The mean height of American men between 20 and 29 years old is 70 inches, and the standard deviation is 2.85 inches. A 20- to 29-year-old man is chosen at random from the population. The probability that he is 6 feet tall or taller is

$$P(72 \le x < \infty) = \int_{72}^{\infty} \frac{1}{2.85\sqrt{2\pi}} e^{-(x-70)^2/6.245}\, dx.$$

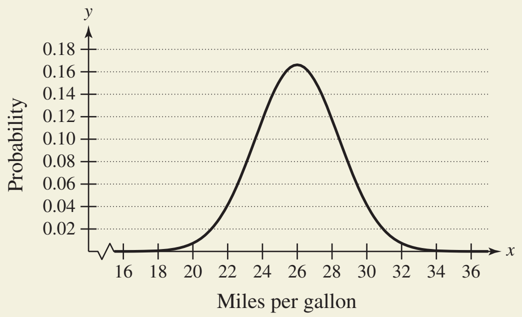
(*Source: National Center for Health Statistics*)

(a) Use a graphing utility to graph the integrand. Use the graphing utility to convince yourself that the area between the $x$-axis and the integrand is 1.

(b) Use a graphing utility to approximate $P(72 \le x < \infty)$.

(c) Approximate $0.5 - P(70 \le x \le 72)$ using a graphing utility. Use the graph in part (a) to explain why this result is the same as the answer in part (b).

**94. HOW DO YOU SEE IT?** The graph shows the probability density function for a car brand that has a mean fuel efficiency of 26 miles per gallon and a standard deviation of 2.4 miles per gallon.




(a) Which is greater, the probability of choosing a car at random that gets between 26 and 28 miles per gallon or the probability of choosing a car at random that gets between 22 and 24 miles per gallon?

(b) Which is greater, the probability of choosing a car at random that gets between 20 and 22 miles per gallon or the probability of choosing a car at random that gets at least 30 miles per gallon?

**Laplace Transforms** **Let $f(t)$ be a function defined for all positive values of $t$. The Laplace Transform of $f(t)$ is defined by**

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

**when the improper integral exists. Laplace Transforms are used to solve differential equations. In Exercises 95–102, find the Laplace Transform of the function.**

**95.** $f(t) = 1$

**96.** $f(t) = t$

**97.** $f(t) = t^2$

**98.** $f(t) = e^{at}$

**99.** $f(t) = \cos at$

**100.** $f(t) = \sin at$

**101.** $f(t) = \cosh at$

**102.** $f(t) = \sinh at$

**103. The Gamma Function** The Gamma Function $\Gamma(n)$ is defined by

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x}\, dx, \quad n > 0.$$

(a) Find $\Gamma(1)$, $\Gamma(2)$, and $\Gamma(3)$.

(b) Use integration by parts to show that $\Gamma(n+1) = n\Gamma(n)$.

(c) Write $\Gamma(n)$ using factorial notation where $n$ is a positive integer.

**104. Proof** Prove that $I_n = \left(\frac{n-1}{n+2}\right) I_{n-1}$, where

$$I_n = \int_0^{\infty} \frac{x^{2n-1}}{(x^2+1)^{n+3}}\, dx, \quad n \ge 1.$$

Then evaluate each integral.

(a) $\displaystyle\int_0^{\infty} \frac{x}{(x^2+1)^4}\, dx$

(b) $\displaystyle\int_0^{\infty} \frac{x^3}{(x^2+1)^5}\, dx$

(c) $\displaystyle\int_0^{\infty} \frac{x^5}{(x^2+1)^6}\, dx$

**105. Finding a Value** For what value of $c$ does the integral

$$\int_0^{\infty} \left(\frac{1}{\sqrt{x^2+1}} - \frac{c}{x+1}\right) dx$$

converge? Evaluate the integral for this value of $c$.

**106. Finding a Value** For what value of $c$ does the integral

$$\int_1^{\infty} \left(\frac{cx}{x^2+2} - \frac{1}{3x}\right) dx$$

converge? Evaluate the integral for this value of $c$.

**107. Volume** Find the volume of the solid generated by revolving the region bounded by the graph of $f$ about the $x$-axis.

$$f(x) = \begin{cases} x \ln x, & 0 < x \le 2 \\ 0, & x = 0 \end{cases}$$

**108. Volume** Find the volume of the solid generated by revolving the unbounded region lying between $y = -\ln x$ and the $y$-axis $(y \ge 0)$ about the $x$-axis.

***u*-Substitution** **In Exercises 109 and 110, rewrite the improper integral as a proper integral using the given $u$-substitution. Then use the Trapezoidal Rule with $n = 5$ to approximate the integral.**

**109.** $\displaystyle\int_0^1 \frac{\sin x}{\sqrt{x}}\, dx, \quad u = \sqrt{x}$

**110.** $\displaystyle\int_0^1 \frac{\cos x}{\sqrt{1-x}}\, dx, \quad u = \sqrt{1-x}$

**111. Rewriting an Integral** Let $\int_{-\infty}^{\infty} f(x)\, dx$ be convergent and let $a$ and $b$ be real numbers where $a \ne b$. Show that

$$\int_{-\infty}^{a} f(x)\, dx + \int_a^{\infty} f(x)\, dx = \int_{-\infty}^{b} f(x)\, dx + \int_b^{\infty} f(x)\, dx.$$

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